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A STATEMENT OF THE PROBLEM OF THERMAL DIAGNOSTICS FOR A "PROTECTIVE" LAYER

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By the problem of diagnostics we understand the task of determining the characteristics of a body from information obtained by experiment about physical fields which arise in it under the influence in a special way of assembled external effects [1, 2]. The operating conditions for a number of articles connected with the combined effect of irradiation, force and thermal loading, strong magnetic fields, and corrosive media necessitate provision of the required safety. Here considerable attention should be devoted to studying the destructive effect of these actions on material characteristics both within the scope of periodic monitoring of the condition of the operating structure, and within a schedule of laboratory tests on the materials developed [3].

In the present work the task is considered of determining changes in the specific heat capacity and thermal conductivity coefficient of a material (as a result of accumulation of microdamage) for a weak curvilinear "protective" layer from the results of thermocouple measurements of temperature at its accessible surface, and at the same time the other surface is subject to the destructive effect of a corrosive medium. The possibility is studied of determining the change in layer thickness (which may also occur as a result of the effect of destructive factors) and clarification of the corrosive medium temperature. In mathematical scheme the problem relates to the type of reverse problems of mathematical physics [4].

1. According to [5] spreading of heat in a nonuniform curvilinear layer $\Omega = \{(x_1, x_2, x_3) | -\infty < x_1, x_2 < \infty, \gamma_1 \leqslant x_3 \leqslant \gamma_2\}$,

$$0 \leqslant \gamma_1 < \gamma_2 \leqslant H - \mathrm{const}, \quad \gamma_1 = \gamma_1(x_1, x_2),$$
$$\gamma_2 = \gamma_2(x_1, x_2, t) \in C^1$$

(Fig. 1) may be described by the relationships

$$C_\varepsilon \theta^{\cdot} - (K\theta_{,i})_{,i} = f; \tag{1.1}$$

$$(a\theta + bn_i\theta_{,i})|_{x_3=\gamma_1} = p(x_1, x_2, t), \tag{1.2}$$
$$n_k = \gamma_{1,k}(1 + \gamma_{1,1}^2 + \gamma_{1,2}^2)^{-1/2}, \quad n_3 = (1 + \gamma_{1,1}^2 + \gamma_{1,2}^2)^{-1/2}, \; k = 1, 2;$$

$$\theta|_{x_3=\gamma_2} = q(x_1, x_2, t); \tag{1.3}$$

$$\theta|_{t=0} = \varphi(x_1, x_2, x_3). \tag{1.4}$$

Here specific heat capacity C_ε and the thermal conductivity coefficient for the layer K depend on spatial variables $x = (x_1, x_2, x_3)$, and relative temperature θ is a function of x and time t. A period means partial derivative with respect to time, the index after a comma is derivative with respect to the corresponding coordinate, and everywhere k = 1, 2, i = 1, 2, 3, m = 1, 2,.... Summing is carried out for a repeated index (if not stipulated to the contrary). Boundary condition (1.2) is the condition of heat exchange at the outer (accessible) surface of the layer $x_3 = \gamma_1$. Condition (1.3) is given at the inner (in contact with some corrosive medium of temperature q) surface $x_3 = \gamma_2$. This surface is assumed to be inaccessible for performing direct measurements, and in view of the effect of various factors its geometry may change with the passage of time, i.e., $\gamma_2 = \gamma_2(x_1, x_2, t)$.

The diagnostic problem considered by us involves determining specific heat capacity $K(x)$ and thermal conductivity coefficient $C_\varepsilon(x)$, and also in clarifying the geometry of the layer [functions $\gamma_2(x_1, x_2, t)$ and corrosive medium temperature $q(x, t)$ from two problems of the form (1.1)-(1.4) with different conditions for initiating thermal processes [after substituting $\{\theta, f, p, \varphi\}^k \to \{\theta, f, p, \varphi\}$ in (1.1)-(1.4)] from supplementary information

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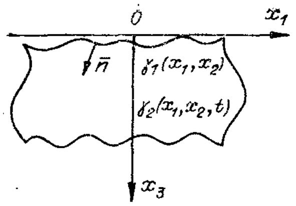

Fig. 1

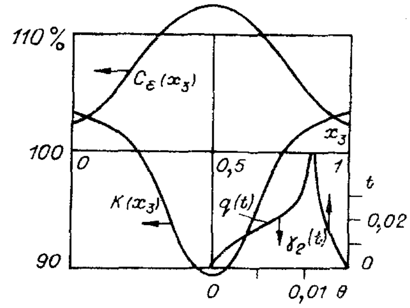

Fig. 2

$$\theta^k|_{x_3=\gamma_1}=\chi^k(x_1,x_2,t),\ K|_{x_3=\gamma_1}=K^{(0)}. \tag{1.5}$$

Relationship (1.5) is a consequence of direct measurements at the outer surface of the layer not subject to destructive effects, then $\gamma_1=\gamma_1(x_1, x_2)$, $K^{(0)}$ = const. Taking account of (1.5) boundary condition (1.2) is substituted by the equation

$$(n_i\theta_{,i}^k)|_{x_3=\gamma_1}=(p^k-a\chi^k)/b=S^k(x_1,x_2,t).$$

We assume that the test layer is weakly inhomogeneous, i.e., values $\|C_\varepsilon-C_\varepsilon^{(0)}\|_{C^0}/C_\varepsilon^{(0)}$, $\|K-K^{(0)}\|_{C^1}/K^{(0)}$ have an order of smallness $O(\varepsilon)$, $0<\varepsilon\ll 1$; where $C_\varepsilon^{(0)}$, $K^{(0)}$ = const are characteristics corresponding to a control (not subject to the effect of destructive factors) uniform layer. This assumption is valid at least in the initial stage of microdamage accumulation in the material. Weak nonuniformity for thermophysical properties of the material does not mean at all that it will be thus in a strength scheme. Thus, α-irradiation of metals changes K by 10-15%, whereas yield strength increases by several factors [6]. This means that nonuniform irradiation causes slight nonuniformity of heat conducting characteristics for metals, and it is possible to estimate the strength reserve of an article by determining these characteristics and separating zones with a different level of radiation damage. By limiting ourselves to considering weakly curvilinear layers we obtain $\|\gamma_1(x_1,x_2)\|_{C^1}/H$, $\|\gamma_2(x_1,x_2,t)-H\|_{C^1}/H\sim O(\varepsilon)$, $0<\varepsilon\ll 1$, $0<H$

We compare the thermal processes $\theta^k(x,t)$ with a process $\theta^{(0)k}(x,t)$ initiated in a similar way which occurs in a control uniform layer $\Omega^{(0)}=\{x|-\infty<x_1,\ x_2<\infty,\ 0\leqslant x_3\leqslant H\}$. The process $\theta^{(0)k}(x,t)$ is described by relationships (1.1)-(1.4) after the substitution $\{C_\varepsilon^{(0)}, K^{(0)}, q^{(0)}, 0, H\}\to\{C_\varepsilon, K, q, \gamma_1, \gamma_2\}$, where $C_\varepsilon^{(0)}$, $K^{(0)}$, $q^{(0)}$, H = const. We assume that the effect of slight nonuniformity, of a small change in the thickness of the protective layer, and small deviation from a constant value $q(x,t)$ on quantitative characteristics of the thermal process excited in the layer is also quite small, i.e., at the accessible surface layer $\theta^k|_{x_3=\gamma_1}=\theta^{(0)k}|_{x_3=\gamma_1}+\varepsilon\theta^{(1)k}|_{x_3=\gamma_1}$, and $\|\theta^{(1)k}|_{x_3=\gamma_1}\|_{C^2}\sim O(\|\theta^{(0)k}|_{x_3=\gamma_1}\|_{C^2})$. We assume that the characteristics sought for the test layer and thermal processes occurring within it are analytical with respect to to the small parameter ε introduced:

$$\{\theta^k, C_\varepsilon, K\}=\sum_{m=0}^{\infty}\varepsilon^m\{\theta^k, C_\varepsilon, K\}.$$

Taking this into account and assuming that $\gamma_1(x_1, x_2)=\varepsilon\alpha_1(x_1, x_2)$, $\gamma_2(x_1,x_2,t)=H+\varepsilon\alpha_2(x_1,x_2,t)$, it is possible by means of expansion into a Taylor series to change over from values of functions at surfaces $x_3=\gamma_1$, $x_3=\gamma_2$ to the corresponding values in planes $x_3 = 0$, $x_3 = H$. The assumptions adopted make it possible according to the perturbation method to change over from (1.1)-(1.5) in region $\Omega\times R_+$ to a chain of equations in region $\Omega^{(0)}\times R_+$

$$\begin{aligned}&C_\varepsilon^{(0)}\theta^{(0)k\cdot}-K^{(0)}\theta_{,ii}^{(0)k}=f^k,\\ &\dots\dots\dots\dots\dots\dots\\ &C_\varepsilon^{(0)}\theta^{(m)k\cdot}-K^{(0)}\theta_{,ii}^{(m)k}=\sum_{s=0}^{m-1}\left[\left(K^{(m-s)}\theta_{,i}^{(s)k}\right)_{,i}-C_\varepsilon^{(m-s)}\theta^{(s)k\cdot}\right],\end{aligned} \tag{1.6}$$

closed by initial and boundary conditions

$$\theta^{(m)k}|_{t=0}=\delta_{0m}\varphi^k; \tag{1.7}$$

$$\sum_{s=0}^{m}\left[\frac{\alpha_1^s}{s!}\frac{\partial^s}{\partial x_3^s}\left(a\theta^{(m-s)k}+b\sum_{r=0}^{m-s}\left(\theta_{,i}^{(m-s-r)k}n_i^{(r)}\right)\right)\right]\Bigg|_{x_3=0}=\delta_{0m}p^k \tag{1.8}$$

($n_i^{(r)}$ are coefficients for expansion of n_i into a series with respect to powers of ε; $n_1^{(0)} = n_2^{(0)} = 0$, $n_3^{(0)} = 1$);

$$\sum_{s=0}^{m} \left(\frac{\alpha_2^s}{s!} \frac{\partial^s}{\partial x_3^s} \theta^{(m-s)k} \right) \Bigg|_{x_3=H} = q^{(m)}; \tag{1.9}$$

$$\sum_{s=0}^{m} \left(\frac{\alpha_1^s}{s!} \frac{\partial^s}{\partial x_3^s} \theta^{(m-s)k} \right) \Bigg|_{x_3=0} = \delta_{0m}\chi^{(0)k} + \delta_{1m}\chi^{(1)k}; \tag{1.10}$$

$$\sum_{s=0}^{m} \left(\frac{\alpha_1^s}{s!} \frac{\partial^s}{\partial x_3^s} K^{(m-s)} \right) \Bigg|_{x_3=0} = \delta_{0m} K^{(0)}. \tag{1.11}$$

In accordance with the physical meaning of the problem characteristics $C_\varepsilon^{(0)}$, $K^{(0)}$, $q^{(0)}$ for the control layer are assumed to be known, i.e., the problem involves clarifying the values sought, which is determining changes in characteristics for the test layer. Structure (1.7)-(1.11) makes it possible to find $\{\theta^k, C_\varepsilon, K\}^{(m)}$ successively with respect to m. We provide an algorithm for finding these values.

2. The first vector equation from (1.6) with m = 0 does not contain unknowns $\{\theta^k, C_\varepsilon, K\}^{(m)}$ (m = 1, 2,...) and together with (1.7)-(1.9) with m = 0 we present with respect to $\theta^{(0)k}$ (k = 1, 2) two normal initial boundary problems for the thermal conductivity equation with constant coefficients in region $\Omega^{(0)} \times R_+$. For simplicity we shall assume that their solution is known and they have a stationary form $\theta^{(0)k}(\mathbf{x}, t) = \exp(-a_k t) g_k(\mathbf{x}) + q_k$ (not summed with respect to k = 1, 2; a_k and q_k are constants), which naturally places limitations on $\{f, p, \varphi\}^k$, i.e., on the condition for initiating thermal processes in the test protective layer.

3. According to (1.6)-(1.11) with respect to functions $\{\theta^k, C_\varepsilon, K\}^{(1)}$ we have

$$C_\varepsilon^{(0)}\theta^{(1)k\cdot} - K^{(0)}\theta_{,ii}^{(1)k} = \left(K^{(1)}\theta_{,i}^{(0)k}\right)_{,i} - C_\varepsilon^{(1)}\theta^{(0)k\cdot}; \tag{3.1}$$

$$\theta^{(1)k}|_{t=0} = 0; \tag{3.2}$$

$$\left(a\theta^{(1)k} + b\theta_{,3}^{(1)k}\right)|_{x_3=0} = -\left(b\theta_{,i}^{(0)k} n_i^{(1)} + \alpha_1\left(a\theta_{,3}^{(0)k} + b\theta_{,33}^{(0)}\right)\right)|_{x_3=0}; \tag{3.3}$$

$$\left(\theta^{(1)k} + \alpha_1\theta_{,3}^{(0)k}\right)|_{x_3=0} = \chi^{(1)k}; \tag{3.4}$$

$$K^{(1)}|_{x_3=0} = 0. \tag{3.5}$$

Determination of these functions is mainly similar to the linearized reverse problem for the thermal conductivity equation considered within the scope of thermoelastic diagnostics in [1, 2]. We only provide the stages of its solution: 1) determination of $\theta^{(1)k}(\mathbf{x}, t)$; 2) reconstruction of $C_\varepsilon^{(1)}(\mathbf{x})$, $K^{(1)}(\mathbf{x})$ from the right-hand part of (3.1).

Stage 1. To (3.1) we apply the operator $l = \partial/\partial t + a_k I$ (I is unit operator), and with each fixed k we obtain

$$C_\varepsilon^{(0)} T^\cdot - K^{(0)} T_{,ii} = 0, \tag{3.6}$$

which we shall consider in region $\{(\mathbf{x}, t) | x_1, x_2 \in R;\ x_3,\ t \in R_+\}$. With respect to the unknown function $T = \theta^{(1)k\cdot} + a_k\theta^{(1)k}$ introduced from (3.2) to (3.4) we have

$$T|_{x_3=0} = l\chi^{(1)}; \tag{3.7}$$

$$T_{,3}|_{x_3=0} = -\frac{a}{b} l\chi^{(1)}. \tag{3.8}$$

Relationships (3.6)-(3.8) are a Cauchy problem for the thermal conductivity equation with data for a provisionally similar surface [7]. The problems pertain to the conditionally correct type which leads to a condition of correctness for the whole diagnostics problem. The uniqueness of its solution in different classes of functions is shown in [4, 7, 8]. Given there are methods for constructing solutions. Thus, the problem may be reduced to a nonhyperbolic Cauchy problem for a wave equation (with data for a provisionally similar surface) by means of transforming L [9], where

$$T(\mathbf{x}, t) = L^{-1}W(\mathbf{x}, t) = (\pi t)^{-1/2} \int_0^\infty \exp(-\tau^2/4t)\, W(\mathbf{x}, \tau)\, d\tau.$$

The problem of determining $W(x, t)$

$$\sqrt{C_\varepsilon^{(0)}} W^{\cdot\cdot} - K^{(0)} W_{,ii} = 0, \quad W|_{x_3=0} = L\ (l\chi^{-1}), \quad W_{,3}|_{x_3=0} = -\tfrac{a}{b} L\,(l\chi^{-1})$$

is also conditionally correct in class of functions C^n [10], but it is classically correct (according to Hadamard) in a class of analytical functions and by means of a Volterra substitution [11] $i\tilde{x}_3 \to t$, $i\tilde{t} \to x_3$ (i is imaginary unit) is reduced to a normal Cauchy problem for a wave equation. Solution of the latter may be found by the Kirchhoff equation [10]. By determining W and then T it is easy from $\theta^{(1)k\cdot} + a_k\theta^{(1)k} = T$, $\theta^{(1)k}|_{t=0} = 0$ to reconstruct $\theta^{(1)k}(x, t) = \exp(-a_k t)\int_0^t \exp(a_k\tau)\,T(x, \tau)\,d\tau$. Here the procedure for stage 1 should be performed with k = 1, 2, i.e., twice.

Stage 2. By knowing $\theta^{(1)k}(x, t)$ it is possible to find the right-hand part of (3.1): $\exp \times (-a_k t) F^{(1)k}(x)$ (not summed with respect to k = 1, 2). Thus, the problem of this stage involves determining $C_\varepsilon^{(1)}(x)$, $K^{(1)}(x)$ from the set of equations

$$a_k C_\varepsilon^{(1)} g_k - (K^{(1)} g_{k,i})_{,i} = F^{(1)k}, \quad x_1, x_2 \in R, \ x_3 \in R_+. \tag{3.9}$$

By excluding from (3.9) unknown function $C_\varepsilon^{(1)}$ which is in these equations in an algebraic way, with respect to $K^{(1)}$ we obtain a linear first order differential equation

$$a_1 g_1 (K^{(1)} g_{2,i})_{,i} - a_2 g_2 (K^{(1)} g_{1,i})_{,i} = a_1 g_1 F^{(1)2} - a_2 g_2 F^{(1)1}. \tag{3.10}$$

The uniqueness of determining $K^{(1)}$ from this equation and from (3.5) (for example it is necessary that plane $x_3 = 0$ is not a characteristic surface nor the envelope of the family of characteristic surfaces [10]) depends on values a_k, $g_k(x)$, i.e., on the conditions for carrying out tests. After finding $K^{(1)}$ from (3.9) we have

$$C_\varepsilon^{(1)} = (F^{(1)1} - (K^{(1)} g_{1,i})_{,i})/a_1 g_1.$$

4. In order to determine $\{C_\varepsilon, K, \theta^{(k)}\}^{(2)}$ from (1.8)-(1.11) we write

$$C_\varepsilon^{(0)}\theta^{(2)k\cdot} - K^{(0)}\theta_{,ii}^{(2)k} = (K^{(2)}\theta_{,i}^{(0)k})_{,i} - C_\varepsilon^{(2)}\theta^{(0)k\cdot} + \left[(K^{(1)}\theta_{,i}^{(1)k})_{,i} - C_\varepsilon^{(1)}\theta^{(1)k\cdot}\right], \tag{4.1}$$

$$\theta^{(2)k}|_{t=0} = 0; \tag{4.2}$$

$$(a\theta^{(2)k} + b\theta_{,3}^{(2)k})|_{x_3=0} = -\left[\sum_{s=1}^{2}\left(\frac{\alpha_1^s}{s}\frac{\partial^s}{\partial x_3^s}\left(a\theta^{(2-s)k} + b\sum_{r=0}^{2-s}(\theta_{,i}^{(2-s-r)}n_i^{(r)})\right)\right)\right]\Bigg|_{x_3=0}; \tag{4.3}$$

$$\theta^{(2)k}|_{x_3=0} = -\left[\alpha_1\theta_{,3}^{(1)k} + \frac{\alpha_1^2}{2}\theta_{,33}^{(0)k}\right]\Bigg|_{x_3=0}; \tag{4.4}$$

$$K^{(2)}|_{x_3=0} = -[\alpha_1 K_{,3}^{(1)}]|_{x_3=0}. \tag{4.5}$$

In (4.1)-(4.5) square brackets separate expressions consisting of functions already found. We present the solution sought in the form $\theta^{(2)k} = U^k + V^k$, where vector function V^k satisfies (4.1)-(4.3) in region $\{(x, t)|x_1, x_2 \in R;\ x_3, t \in R_+\}$ in which as the right-hand parts there are only expressions in square brackets. The problem of determining V^k in a mathematical scheme is two normal initial boundary problems which describe processes of spreading of heat in a half-space with prescribed heat exchange conditions at the boundary and taking account of distributed heat sources. After finding $V^k(x, t)$ determination of U^k, $C_\varepsilon^{(2)}$, $K^{(2)}$ from (4.1)-(4.5) is entirely similar to determining $\theta^{(1)k}$, $C_\varepsilon^{(1)}$, $K^{(1)}$ from (3.1)-(3.5). Thus, it is possible to obtain all of the expansion terms for the characteristics of thermal processes $\theta^k(x, t)$ used for diagnostics and expansion terms for the characteristics of the test layer $C_\varepsilon(x)$, $K(x)$.

5. In solving the problem for functions sought $\theta^k(x, t)$, $C_\varepsilon(x)$, $K(x)$ were determined in half-space $x_3 \geqslant 0$. Only the limitations of these functions in the region occupied by the layer have a physical meaning. However, the shape of the layer inner surface in contact with a corrosive medium (in contrast to the accessible surface) is unknown and its determination (clarification) apart from clarifying the value of temperature q at its surface is also of practical interest. In some cases these values are found from analyzing temperature fields $\theta^k(x, t)$. We make use of the fact that q is the temperature of corrosive medium and it does not depend on conditions for exciting thermal processes used in carrying out tests. This means that at the surface sought $\theta^k|_{x_3=\nu_2} = q$, $k = 1, 2$. These relationships may be considered

as a set of equations with respect to $\gamma_2(x_1, x_2, t)$ and $q(x_1, x_2, t)$. In contrast to the problem of determining $C_\varepsilon(\mathrm{x})$, $K(\mathrm{x})$, where classical and conditional correctness entering into the problem make all of it conditionally correct, in finding γ_2, q is not possible without applying additional conditions to guarantee either existence or uniqueness of the solution. However, sometimes this information of practical importance may be obtained quite easily from analyzing numerical values of temperature fields θ^1 and θ^2.

We write some results of calculations. In (1.6)-(1.11) dimensionless substitution of variables was carried out by the equations $\bar{x}_i = x_i/H$, $\bar{t} = K^{(0)}t/(C_\varepsilon^{(0)}H^2)$, $\bar{\theta} = \theta/T_0$, $\bar{C}_\varepsilon = C_\varepsilon/C_\varepsilon^{(0)}$, $\bar{K} = K/K^{(0)}$. Within the scope of the illustrative example it was assumed that $\gamma_1 \equiv 0$, and γ_2, C_ε, K did not depend on x_1, x_2. Then use of thermal effects distributed uniformly over surface $x_3 = 0$ and heat sources independent of x_1, x_2 gives the diagnostic problem for a protective layer unidimensional with respect to spatial coordinate x_3.

For the calculations use was made of $a_1 = 0$, $a_2 = 4\pi^2$, $\bar{K}|_{x_3=0} = 1, 3$, $\bar{g}_1 = \sin \pi x_3$, $\bar{g}_2 = -\sin \pi \bar{x}_3$, $\bar{\chi}^{(1)1} = \bar{\chi}^{(1)2} = 0$, $S^1 = 0.1(1 - \exp(-9\pi^2\bar{t}))$, $S^2 = 0.4(\exp(-4\pi^2\bar{t}) - \exp(-\pi^2\bar{t}))$.

Given in Fig. 2 is the distribution of $C_\varepsilon(x_3)$, $K(x_3)$ in a percentage ratio to their control constant values and also curves for the change in layer thickness $\gamma_2(t)$ and corrosive medium temperature q(t) with the passage of time. The values provided correspond to considering the first three terms of expansions. Consideration of the fourth term does not make a marked quantitative contribution in calculating these values, and this may point to the convergence of the series.

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